Analytic solutions of some coupled nonlinear equations

F. T. Hioe
Department of Physics, St. John Fisher College, Rochester, New York 14618
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New analytic solutions for coupled nonlinear Schrödinger, coupled nonlinear equations of higher order, coupled Korteweg–de Vries, and coupled Boussinesq equations are presented. Applications to solitary waves, e.g., in birefringent optical fibers, are discussed. [S1063-651X(97)00612-0]

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I. INTRODUCTION

The problem of integrability and nonintegrability of a Hamiltonian system with two degrees of freedom has been a subject of considerable interest for many years [1]. The second invariants, and methods for their search, for many two-dimensional Hamiltonian systems, were summarized and given by Hietarinta [2]. When a system is known to be integrable, some, but not necessarily all, analytic expressions for the time evolution of the two spatial coordinates x and y have been found in some cases.

Interests in finding specific analytic solutions for $x(t)$ and $y(t)$ have come from another field in physics: their applications to the problem of finding periodic solitary waves in birefringent optical fibers. The slowly varying complex components or envelopes $\phi_m(z,t)$, $m=1,2$, of the electric fields of the two mutually orthogonal polarizations propagating along the z axis satisfy the following pair of coupled nonlinear Schrödinger-like equations [3]:

$$i\phi_{1z}+\phi_{1tt}+\kappa\phi_1+p(|\phi_1|^2+|\phi_2|^2)\phi_1+q(\phi_1^2+\phi_2^2)\phi_1^\star=0,$$

$$i\phi_{2z}+\phi_{2tt}-\kappa\phi_2+p(|\phi_1|^2+|\phi_2|^2)\phi_2+q(\phi_1^2+\phi_2^2)\phi_2^\star=0, \tag{1.1}$$

where p and q are dimensionless parameters characteristic of the medium that satisfy the relation $p+q=1$, κ is related to the birefringence of the fiber, and the subscripts 1 and 2 for ϕ are to be distinguished from the subscripts in z and t, which denote derivatives with respect to z and t, respectively. Christodoulides and Joseph [4] and Florjanczyk and Tremblay [5] showed that we may first search for the stationary-wave solution of the form

$$\phi_1(z,t)=x(t)\exp(i\Omega z),$$

$$\phi_2(z,t)=y(t)\exp(i\Omega z), \tag{1.2}$$

where Ω is a real constant, and $x(t)$ and $y(t)$ are (real) functions of t only. Equations (1.1) are shown to reduce to the following, which we shall call the *dynamical* coupled nonlinear Schrödinger equations:

$$\ddot{x}-Ax+(x^2+y^2)x=0,$$

$$\ddot{y}-By+(x^2+y^2)y=0, \tag{1.3}$$

where $A=\Omega-\kappa$ and $B=\Omega+\kappa$. Equations (1.3) are the dynamical equations of motion for a system whose Hamiltonian is

$$H=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-\tfrac{1}{2}(Ax^2+By^2)+\tfrac{1}{4}(x^2+y^2)^2, \tag{1.4}$$

where $\dot{x}$ denotes dx/dt, etc. The Hamiltonian system (1.4) is usually referred to as a coupled quartic oscillator system. From the nontraveling waves (1.2), Florjanczyk and Trembley [5] showed that traveling waves can be constructed from

$$\phi_1'(z,t)=\phi_1(z,t-z/v)\exp[i(t-z/2v)/(2v)],$$

$$\phi_2'(z,t)=\phi_2(z,t-z/v)\exp[i(t-z/2v)/(2v)], \tag{1.5}$$

where v is the velocity of the waves, and that they satisfy Eqs. (1.1). Following the work of Florjanczyk and Trembley who presented an analytic periodic wave solution for Eqs. (1.3), Kostov and Uzunov [6] presented three additional periodic solutions of Eqs. (1.3).

In this paper, we present three new periodic wave solutions of Eqs. (1.3) in Sec. II. We also studied other coupled dynamical systems, and present periodic wave solutions for a special case of coupled nonlinear equations of higher order in Sec. III, three periodic wave solutions for coupled Korteweg–de Vries (KdV) equations in Sec. IV and for coupled Boussinesq equations in Sec. V. Solitary wave solutions for the corresponding coupled equations are also presented.

II. COUPLED NONLINEAR SCHRÖDINGER (CNLS) EQUATIONS

As we mentioned above, four periodic wave solutions have been found previously for the dynamical CNLS equations (1.3). Because the regimes for which the solutions apply were not explicitly given in some cases, we shall present them here, with the specifications for the regimes in which they apply, with the three new solutions we found. The seven solutions, numbered (I) to (VII), are presented in the order that appears to be most natural; the first three are ''single'' solutions in which each of the $x(t)$ and $y(t)$ is expressed in terms of a single Jacobian elliptic function (of modulus k), while the remaining four are ''product'' solutions in which each of the $x(t)$ and $y(t)$ is expressed in terms of product of two Jacobian elliptic functions. Solution (V) was given by Florjanczyk and Trembley [5] and solutions (I), (IV), and (VI) were given by Kostov and Uzunov [6]. Solutions (II),

TABLE I. Three solutions (A), (B), and (C) for coupled NLS, KdV, and Boussinesq equations.

	(A)	(B)	(C)
ε	$+1$	$+1$	-1
c_1	<0	<0	>0
b_1	>0	>0	<0
b_2	<0	<0	<0
	$\lvert b_1\rvert\geqslant 2\lvert b_2\rvert$	$2\lvert b_2\rvert\geqslant\lvert b_1\rvert\geqslant\frac{1}{2}\lvert b_2\rvert$	$\lvert b_2\rvert\geqslant\lvert b_1\rvert$
k^2	$\frac{\lvert b_1\rvert+\lvert b_2\rvert}{2\lvert b_1\rvert-\lvert b_2\rvert}$	$\frac{2\lvert b_1\rvert-\lvert b_2\rvert}{\lvert b_1\rvert+\lvert b_2\rvert}$	$\frac{2\lvert b_1\rvert+\lvert b_2\rvert}{\lvert b_1\rvert+2\lvert b_2\rvert}$
α	$\frac{1}{2}\sqrt{\frac{1}{3}(2\lvert b_1\rvert-\lvert b_2\rvert)}$	$\frac{1}{2}\sqrt{\frac{1}{3}(\lvert b_1\rvert+\lvert b_2\rvert)}$	$\frac{1}{2}\sqrt{\frac{1}{3}(\lvert b_1\rvert+2\lvert b_2\rvert)}$
f_1	$\mathrm{sn}(\alpha\xi,k)$	$k\ \mathrm{sn}(\alpha\xi,k)$	$(k/k')\mathrm{cn}(\alpha\xi,k)$
f_2	$\mathrm{cn}(\alpha\xi,k)$	$\mathrm{dn}(\alpha\xi,k)$	$(1/k')\mathrm{dn}(\alpha\xi,k)$

(III), and (VII) are the new ones. In all the solutions presented, we assume $B>A>0$, and more restrictive conditions apply for some.

(I) $x=C_1\ \mathrm{sn}(\alpha t,k)$, $y=C_2\ \mathrm{cn}(\alpha t,k)$, where $\alpha^2=(B-A)/k^2$, $C_1^2=\alpha^2-2\alpha^2k^2+B$, $C_2^2=\alpha^2+\alpha^2k^2+A$; it is valid in the regime where $C_1^2<C_2^2$, $C_1^2\leqslant B$, $C_2^2\geqslant A$, and $C_2^2-A\geqslant 2(-C_1^2+B)$.

(II) $x=C_1k\ \mathrm{sn}(\alpha t,k)$, $y=C_2\ \mathrm{dn}(\alpha t,k)$, where $\alpha^2=B-A$, $C_1^2=-2\alpha^2+\alpha^2k^2+B$, $C_2^2=\alpha^2+\alpha^2k^2+A$; it is valid in the regime where $C_1^2<C_2^2$, $C_1^2\leqslant B$, $C_2^2\geqslant A$, $2(-C_1^2+B)\geqslant C_2^2-A\geqslant\frac{1}{2}(-C_1^2+B)$.

(III) $x=C_1(k/k')\mathrm{cn}(\alpha t,k)$, $y=C_2(1/k')\mathrm{dn}(\alpha t,k)$, where $\alpha^2=(B-A)/k'^2$, $C_1^2=2\alpha^2-\alpha^2k^2-B$, $C_2^2=\alpha^2-2\alpha^2k^2+A$, and where k' is the complementary modulus; it is valid in the regime where $C_2^2\leqslant A$.

(IV) $x=Ck\ \mathrm{sn}(\alpha t,k)\mathrm{cn}(\alpha t,k)$, $y=C\ \mathrm{cn}(\alpha t,k)\mathrm{dn}(\alpha t,k)$, where $C^2=2(4B-A)/5$, $\alpha^2=(B-A)/3$, $k^2=(4B-A)/[5(B-A)]$; it is valid for $B\geqslant 4A$.

(V) $x=C\ \mathrm{sn}(\alpha t,k)\mathrm{dn}(\alpha t,k)$, $y=C\ \mathrm{cn}(\alpha t,k)\mathrm{dn}(\alpha t,k)$, where $C^2=2(4B-A)/5$, $\alpha^2=(4B-A)/15$, $k^2=5(B-A)/(4B-A)$; it is valid for $A<B\leqslant 4A$.

(VI) $x=C\alpha^2k^2\ \mathrm{sn}(\alpha t,k)\mathrm{cn}(\alpha t,k)$, $y=C\alpha^2\ \mathrm{dn}^2(\alpha t,k)+C_1$, where $C^2=18/(B-A)$, $C_1=C(-B+4A)/5$, $\alpha^2=(1/10)[2B-3A+\sqrt{\frac{5}{3}(B^2-A^2)}]$,

$$k^2=\frac{2\sqrt{\frac{5}{3}(B^2-A^2)}}{\sqrt{\frac{5}{3}(B^2-A^2)}+2B-3A};$$

it is valid for $B\geqslant 4A$.

(VII) $x=C\alpha^2k\ \mathrm{sn}(\alpha t,k)\mathrm{dn}(\alpha t,k)$, $y=C\alpha^2\ \mathrm{dn}^2(\alpha t,k)+C_1$, where $C^2=18/(B-A)$,

$$C_1=\frac{C}{2}\left[\frac{1}{5}\sqrt{\tfrac{5}{3}(B^2-A^2)}+\frac{1}{3}(B-A)\right],$$

$$\alpha^2=\frac{1}{5}\sqrt{\tfrac{5}{3}(B^2-A^2)},\quad k^2=\frac{\sqrt{\frac{5}{3}(B^2-A^2)}+2B-3A}{2\sqrt{\frac{5}{3}(B^2-A^2)}}.$$

It is valid in the regime where $8A/7\leqslant B\leqslant 4A$.

For Eqs. (1.3), besides the Hamiltonian (1.4), which is the first invariant, the second invariant [7] is $I=(x\dot{y}-\dot{x}y)^2+(A-B)(2\dot{x}^2-2Ax^2+x^4+x^2y^2)$.

The seven solutions for x and y thus provide seven solitary wave solutions for Eqs. (1.1). As shown by Florjanczyk and Trembley [5], the following substitutions

$$\Phi_1=(\phi_1+i\phi_2)/\sqrt{2},\quad \Phi_2=(\phi_1-i\phi_2)/\sqrt{2},$$

$$z=(\gamma+1)Z/2,\quad t=\sqrt{\gamma+1}T,\tag{2.1}$$

$$q=(\gamma-1)/(\gamma+1),\quad \kappa=2\sigma/(\gamma+1),$$

transform Eqs. (1.1) into

$$i\Phi_{1Z}+\tfrac{1}{2}\Phi_{1TT}+\sigma\Phi_2+(|\Phi_1|^2+\gamma|\Phi_2|^2)\Phi_1=0,$$
$$i\Phi_{2Z}+\tfrac{1}{2}\Phi_{2TT}+\sigma\Phi_1+(|\Phi_2|^2+\gamma|\Phi_1|^2)\Phi_2=0.\tag{2.2}$$

For the special case of $\gamma=1$, we can use the transformation suggested by Bélanger and Paré [8],

$$\Phi=\Psi_1(z,t)\cos(\sigma z)+i\Psi_2(z,t)\sin(\sigma z),$$

$$\Psi=\Psi_2(z,t)\cos(\sigma z)+i\Psi_1(z,t)\sin(\sigma z),\tag{2.3}$$

to transform Eqs. (2.2) into the standard *symmetric* CNLS equations given by

$$i\Psi_{1z}+\Psi_{1tt}+(|\Psi_1|^2+|\Psi_2|^2)\Psi_1=0,$$

$$i\Psi_{2z}+\Psi_{2tt}+(|\Psi_1|^2+|\Psi_2|^2)\Psi_2=0.\tag{2.4}$$

In contrast to the known solution [9] of Eqs. (2.4), we present here new *superposition* solutions for which (a) each of the Ψ_1 and Ψ_2 is expressed in terms of the sum of two solitary waves $f_1(\xi)$ and $f_2(\xi)$, where

$$\xi=2(t-z/v),\tag{2.5}$$

and v is the velocity of the waves, and (b) there are three different pairs of elliptic functions for $f_1(\xi)$ and $f_2(\xi)$ applicable to various regimes. A factor of 2 has been inserted for the definition of ξ in Eq. (2.5) because Table I to be presented will be shown to be applicable also for other dynamical systems for which ξ is defined differently. Specifically, the superposition solutions are expressed by

$$\Psi_m=\sum_{n=1}^{2} C_{mn}f_n(\xi)\exp\left[i\left(K_nz+\frac{t}{2v}\right)\right], \quad m=1,2, \tag{2.6}$$

where the four amplitudes C_{mn} are required to satisfy

$$C_{11}C_{12}+C_{21}C_{22}=0, \tag{2.7}$$

but none of them needs to be equal to zero generally. Let us define

$$b_1=-K_1-\frac{1}{4v^2}+C_{12}^2+C_{22}^2,$$

$$b_2=-K_2-\frac{1}{4v^2}+\varepsilon(C_{11}^2+C_{21}^2),$$

$$c_1=C_{11}^2+C_{21}^2-\varepsilon(C_{12}^2+C_{22}^2), \tag{2.8}$$

where ε can be equal to $+1$ or -1. The three different pairs of elliptic functions for $f_1(\xi)$ and $f_2(\xi)$ are given, in columns marked (A), (B), and (C), together with the conditions that specify the regimes in which they are applicable, in Table I. A condition

$$\varepsilon|b_1|+|b_2|=\tfrac{3}{2}|c_1| \tag{2.9}$$

must be satisfied for any one of the solutions (A), (B), or (C), which imposes another condition, besides Eq. (2.7), on the amplitudes of the waves. Substituting the $f_1(\xi)$ and $f_2(\xi)$ from Table I into Eq. (2.6), and substituting Eq. (2.6) into Eqs. (2.3) give three new solutions of Eqs. (2.2) *for the special case of* $\gamma=1$.

III. A SPECIAL CASE OF COUPLED NONLINEAR EQUATIONS OF HIGHER ORDER

Instead of Eq. (1.4) for a coupled quartic oscillator system, let us consider a circularly symmetric coupled sextic oscillator system whose Hamiltonian is

$$H=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-\tfrac{1}{2}A(x^2+y^2)+\tfrac{1}{6}(x^2+y^2)^3 \tag{3.1}$$

for which the coupled equations of motion are

$$\ddot{x}-Ax+(x^2+y^2)^2x=0,$$
$$\ddot{y}-Ay+(x^2+y^2)^2y=0. \tag{3.2}$$

Since H can be expressed in $r=(x^2+y^2)^{1/2}$ only, it is clearly an integrable system. We have found a simple analytic periodic solution of Eqs. (3.2) for the case $A>0$; it is $x=\{C[\mathrm{dn}(\alpha t,k)+\mathrm{cn}(\alpha t,k)]\}^{1/2}$, $y=\{C[\mathrm{dn}(\alpha t,k)-\mathrm{cn}(\alpha t,k)]\}^{1/2}$, where $\alpha^2=4A/(2-k^2)$, $C^2=3A/[4(2-k^2)]$, $0\leqslant k^2\leqslant 1$. The second invariant is obviously the angular momentum $I=x\dot{y}-\dot{x}y$, which for our solution is equal to $C\alpha k'$.

Applications of the above result can be seen from the following example. Suppose two real field components $\phi_1(z,t)$ and $\phi_2(z,t)$, which propagate along the z axis satisfy the following coupled nonlinear equations:

$$\phi_{1zz}+\beta\phi_{1tt}-A\phi_1+(\phi_1^2+\phi_2^2)^2\phi_1=0,$$
$$\phi_{2zz}+\beta\phi_{2tt}-A\phi_2+(\phi_1^2+\phi_2^2)^2\phi_2=0. \tag{3.3}$$

A solitary wave solution of Eq. (3.3) where $\phi_1(\xi)$ and $\phi_2(\xi)$ depend on $\xi=z-vt$, v being the velocity of the waves, can be easily deduced. It is $\phi_1=\{C[\mathrm{dn}(\gamma\xi,k)+\mathrm{cn}(\gamma\xi,k)]\}^{1/2}$, $\phi_2=\{C[\mathrm{dn}(\gamma\xi,k)-\mathrm{cn}(\gamma\xi,k)]\}^{1/2}$, where $\gamma^2=4A/[(2-k^2)(1+\beta v^2)]$, $C^2=3A/[4(2-k^2)]$, $0\leqslant k^2\leqslant 1$.

IV. COUPLED KORTEWEG–DE VRIES EQUATIONS

Consider the dynamical coupled KdV equations given by

$$\dddot{x}_1+\beta\dot{x}_1+R(x_1+x_2)\dot{x}_1=0,$$
$$\dddot{x}_2+\beta\dot{x}_2+R(x_1+x_2)\dot{x}_2=0, \tag{4.1}$$

where β and R are real constants and where for later notational purposes, we have replaced x and y by x_1 and x_2, respectively. Our analytic periodic solutions are expressed in the form

$$x_m=\sum_{m=1}^{2} C_{mn}f_n^2, \quad m=1,2, \tag{4.2}$$

where the C's are constants and $f_1(t)$ and $f_2(t)$ will be given in terms of Jacobian elliptic functions. First let us define

$$b_1=(C_{12}+C_{22})R+\beta, \tag{4.3a}$$

$$b_2=\varepsilon(C_{11}+C_{21})R+\beta, \tag{4.3b}$$

$$c_1=[C_{11}+C_{21}-\varepsilon(C_{12}+C_{22})]R, \tag{4.3c}$$

Three different pairs of elliptic functions $f_1(t)$ and $f_2(t)$ for Eqs. (4.2) are given, together with the conditions that specify the regimes in which they are applicable, in Table I, where ξ is replaced by t.

The corresponding coupled KdV equations for $\phi_1(z,t)$ and $\phi_2(z,t)$ are

$$\phi_{1zzz}+\beta'\phi_{1t}+R(\phi_1+\phi_2)\phi_{1z}=0,$$
$$\phi_{2zzz}+\beta'\phi_{2t}+R(\phi_1+\phi_2)\phi_{2z}=0. \tag{4.4}$$

The solitary wave solutions of Eq. (4.4) for which $\phi_1(\xi)$ and $\phi_2(\xi)$ depend on

$$\xi=z-vt \tag{4.5}$$

only, where v is the velocity of the waves, can be written down as in Eqs. (4.2). The solutions of Eqs. (4.4) are obtained by replacing $x_m(t)$ by $\phi_m(\xi)$, $f_n(t)$ by $f_n(\xi)$, in Eqs. (4.2), where ξ is given by Eq. (4.5), i.e.,

$$\phi_m=\sum_{n=1}^{2} C_{mn}f_n^2. \tag{4.6}$$

Instead of Eqs. (4.3), the b's and c are here defined by

$$b_1=(C_{12}+C_{22})R-\beta' v, \tag{4.7a}$$

$$b_2=\varepsilon(C_{11}+C_{21})R-\beta' v, \tag{4.7b}$$

$$c_1=[C_{11}+C_{21}-\varepsilon(C_{12}+C_{22})]R. \tag{4.7c}$$

Table I gives three solutions of Eq. (4.6) for Eq. (4.4). Note that ξ is defined here by Eq. (4.5), which is different from the ξ defined by Eq. (2.5) for the CNLS equations.

V. COUPLED BOUSSINESQ EQUATIONS

Consider the dynamical coupled Boussinesq equations given by

$$\begin{aligned} x_{1tttt}+\beta x_{1tt}+R[(x_1+x_2)x_{1t}]_t=0,\\ x_{2tttt}+\beta x_{2tt}+R[(x_1+x_2)x_{2t}]_t=0. \end{aligned} \tag{5.1}$$

As for the dynamical coupled KdV equations, we found analytic periodic solutions of Eqs. (5.1) in the form expressed also by Eqs. (4.2). With the b's and c defined as in Eqs. (4.3) where β and R are now constants appearing in Eqs. (5.1), the same three pairs of elliptic functions $f_1(t)$ and $f_2(t)$ given in Table I (where ξ is replaced by t) substituted into Eqs. (4.2) give solutions of Eqs. (5.1), for the three regimes in which they are applicable.

The corresponding coupled Boussinesq equations are

$$\begin{aligned} \phi_{1zzzz}+\alpha'\phi_{1zz}+\beta'\phi_{1tt}+R[(\phi_1+\phi_2)\phi_{1z}]_z=0,\\ \phi_{2zzzz}+\alpha'\phi_{2zz}+\beta'\phi_{2tt}+R[(\phi_1+\phi_2)\phi_{2z}]_z=0. \end{aligned} \tag{5.2}$$

Solitary wave solutions of Eqs. (5.2) in which $\phi_1(\xi)$ and $\phi_2(\xi)$ depend on $\xi=z-vt$ only [the same definition of ξ as defined by Eq. (4.5)] can be expressed as in Eqs. (4.6). Instead of Eqs. (4.7), the b's and c are here defined by

$$b_1=(C_{12}+C_{22})R+\alpha'+\beta' v^2, \tag{5.3a}$$

$$b_2=\varepsilon(C_{11}+C_{21})R+\alpha'+\beta' v^2, \tag{5.3b}$$

$$c_1=[C_{11}+C_{21}-\varepsilon(C_{12}+C_{22})]R. \tag{5.3c}$$

With these definitions of b's and c, Eqs. (4.6) and Table I [where ξ is defined by Eq. (4.5)] give three solutions of Eqs. (5.2).

Our solutions reveal a great deal of similarity between the solutions of coupled KdV and coupled Boussinesq equations.

Finally, we remark that (i) the prospect that shape-preserving Jacobian elliptic ''pulse trains'' in optical fibers can be produced and observed appears to have become a real possibility following the first experimental observation of the evolution of an arbitrarily shaped input optical pulse-train to the shape-preserving Jacobian elliptic pulse-train corresponding to the Maxwell-Bloch equations, as reported recently by Shultz and Salamo [10]; and (ii) the possibility of controlling the pulse propagation by varying the amplitudes (the C's in the solutions presented) of the coupled waves [11] should find practical applications in the near future.

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